Quantum factoring, discrete logarithms and the hidden subgroup problem

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Abstract

Amongst the most remarkable successes of quantum computation are Shor's efficient quantum algorithms for the computational tasks of integer factorisation and the evaluation of discrete logarithms. In this article we review the essential ingredients of these algorithms and draw out the unifying generalization of the so-called abelian hidden subgroup problem. This involves an unexpectedly harmonious alignment of the formalism of quantum physics with the elegant mathematical theory of group representations and fourier transforms on finite groups. Finally we consider the non-abelian hidden subgroup problem mentioning some open questions where future quantum algorithms may be expected to have a substantial impact.

1 Introduction

Quantum algorithms exploit quantum physical effects to provide new modes of computation which are not available to "conventional" (classical) computers. In some cases these modes provide efficient (i.e. polynomial time) algorithms for computational tasks where no efficient classical algorithm is known. The most celebrated quantum algorithm to date is Shor's algorithm for integer factorisation [7, 8, 10]. It provides a method for factoring any integer of n digits in time (i.e. in a number of computational steps) that grows less rapidly than $O(n^3)$. Thus it is a polynomial time algorithm in contrast to the best known classical algorithm for this fundamental problem, which runs in superpolynomial time of order $\exp(n^{\frac{1}{3}}(\log n)^{\frac{2}{3}})$.

At the heart of the quantum factoring algorithm is the discrete Fourier transform and the remarkable ability of a quantum computer to efficiently determine periodicities. This in turn rests on the mathematical formalism of fast Fourier transforms combined with principles of quantum physics. In this article we will review these issues including further applications such as the evaluation of discrete logarithms. We will outline a unifying generalization of these ideas: the so-called hidden subgroup problem which is just a natural group theoretic generalization of the problem of periodicity determination. Finally we will consider some interesting open questions related to the hidden subgroup problem for non-abelian groups, where future quantum algorithms may be expected to have a substantial impact.

We may think of periodicity determination as a particular kind of pattern recognition. Quantum computers are able to store and process large volumes of information, represented compactly in the identity of an entangled quantum state, but quantum measurement theory severely restricts our access to the information. Indeed only a relatively small amount of the information may be read out but this may be of a "global" nature, such as a few broad features of a large intricate pattern, which may be impossible to extract efficiently by classical means. This intuition is exemplified in the earliest quantum algorithm, known as Deutsch's algorithm [10]. Here we are given a black box that computes a Boolean function of n variables (i.e. a function of all n bit strings with one-bit values). It is promised that the function is either a constant function or 'balanced' in the sense that exactly half of the values are 0 and

half are 1. We wish to determine with certainty whether the given function is balanced or constant, using the least number of queries to the box. Thus we are asking for one bit of information about the 2^n values of the function. Classically $2^{n-1} + 1$ queries are necessary in the worst case (if the problem is to be solved with *certainty*) but quantumly the problem can be solved in all cases with just one query [10]. However if we tolerate any arbitrarily small probability of error in the answer then there is also a classical algorithm using only a constant number of queries.

Inspired by these results, Simon [6] considered a more complicated situation of a class of functions from n bits to n bits and developed a computational task displaying an exponential gap between the classical and quantum query complexities, even if (in contrast to Deutsch's algorithm) the algorithm is required to work only with bounded error probability of 1/3 i.e. we allow probabilistic algorithms and in any run the answer must be correct with probability at least 2/3.

In retrospect (c.f. below) Simon's problem turns out to be an example of a "generalized periodicity" or hidden subgroup problem, for the group of n bit strings under binary bitwise addition. Shor recognized the connection with periodicity determination and generalized the constructions to the group of integers modulo N, showing significantly that the associated discrete Fourier transform may be efficiently implemented in that context as well. Finally using known reductions of the tasks of integer factorisation and evaluation of discrete logarithms to periodicity determinations, he was able to give polynomial time quantum algorithms for these computational tasks too.

2 The quantum Fourier transform and periodicities

We begin with an account of how a quantum computer may efficiently determine the periodicity of a given periodic function. Consider the following basic example. Suppose that we have a black box which computes a function $f: \mathcal{Z}_N \to \mathcal{Z}$ that is guaranteed to be periodic with some period r:

$$f(x+r) = f(x)$$
 for all x (1)

Here \mathcal{Z}_N denotes the additive group of integers modulo N. We also assume that f does not take the same value twice within any single period. Note that eq. (1) can hold only if r divides N exactly.

Our aim is to determine r. Classically (in the absence of any further information about f) we can merely try different values of x in the black box hoping for two equal results which will then give information about r. Generally we will require O(N) random tries to hit two equal values with high probability. Using quantum effects we will be able to find r using only $O((\log N)^2)$ steps, which represents an exponential speedup over any known classical algorithm.

In the quantum context we assume the black box is a coherent quantum process which evolves the input state $|x\rangle |0\rangle$ to $|x\rangle |f(x)\rangle$ i.e. the values of x and f(x) are labels on a suitable set of orthogonal states. We begin by computing all values of f in equal superposition, using one application of the box. To do this we set up the input register in the equal superposition $\frac{1}{\sqrt{N}} \sum_{x} |x\rangle$, apply the function and obtain the state:

$$|f\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle |f(x)\rangle \tag{2}$$

Although the description of this state embodies all the values of f and hence the periodicity, it is not immediately clear how to extract the information of r! If we measure the value in the second register, giving a value y_0 say, then the state of the first register will be reduced to an equal superposition of all those $|x\rangle$'s such that $f(x) = y_0$. If x_0 is the least such x and N = Kr then we will obtain in the first register the periodic state

$$|\psi\rangle = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} |x_0 + kr\rangle \tag{3}$$

It is important to note here that $0 \le x_0 \le r-1$ has been generated at random, corresponding to having seen any value y_0 of f with equal probability. So if we now measure the value in this register, the overall result is merely to produce a number between 0 and N-1 uniformly at random, giving no information at all about the value of r.

The resolution of this difficulty is to use the Fourier transform which, even for classical data, is known to be able to pick out periodic patterns in a set of data regardless of how the whole pattern is shifted. The discrete Fourier transform \mathcal{F} for integers modulo N is the N by N unitary matrix with entries

$$\mathcal{F}_{ab} = \frac{1}{\sqrt{N}} e^{2\pi i \frac{ab}{N}} = \frac{1}{\sqrt{N}} \chi_a(b) \tag{4}$$

where we have introduced the functions

$$\chi_l(m) = \exp 2\pi i \frac{lm}{N}.\tag{5}$$

If we apply this unitary transform to the state $|\psi\rangle$ above then we obtain [8]

$$\mathcal{F}|\psi\rangle = \frac{1}{\sqrt{r}} \sum_{i=0}^{r-1} e^{2\pi i \frac{x_0 j}{r}} \left| j \frac{N}{r} \right\rangle \tag{6}$$

Indeed a direct calculation shows that the labels which appear with non-zero amplitude are those values of l satisfying

$$\chi_l(r) = e^{2\pi i \frac{lr}{N}} = 1 \tag{7}$$

i.e. lr is a multiple of N and furthermore they appear with equal squared amplitudes. This calculation uses the periodic structure of eq. (3) and the elementary identity

$$\sum_{k=0}^{K-1} \left(e^{2\pi i \frac{l}{K}} \right)^k = \begin{cases} 0 \text{ if } l \text{ is not a multiple of } K \\ K \text{ if } l \text{ is a multiple of } K \end{cases}$$
 (8)

It is important to note here that the random shift x_0 no longer appears in the ket labels. If we now read the label we will obtain a value c say, which is necessarily a multiple of N/r i.e. $c = \lambda N/r$. Thus we can write

$$\frac{c}{N} = \frac{\lambda}{r} \tag{9}$$

where c and N are known numbers and and $0 \le \lambda \le r - 1$ has been chosen uniformly at random by the measurement. Now if the randomly chosen λ is fortuitously coprime to r (i.e. λ and r have no common factors) we can determine r by cancelling c/N down to an irreducible fraction. What is the probability that a randomly chosen r actually is coprime

to r? According to a basic theorem of number theory (c.f. [1, 2] and appendix A of [8]), the number of co-primes less than r goes as $e^{-\gamma}r/\log\log r$ (where γ is Euler's constant) for large r. Thus the probability that our randomly chosen λ is coprime to r is $O(1/\log\log r)$ which exceeds $O(1/\log\log N)$. Hence if we repeat the above procedure $O(\log\log N)$ times we can succeed in determining r with any prescribed probability $1 - \epsilon$ as close to 1 as desired.

We noted above that we want our quantum algorithm to run in time poly(log N) i.e. in a number of steps which is polynomial in $\log N$ rather than N itself, to achieve an exponential speed up over any known classical algorithm for determining periodicity. We showed above that merely $O(\log \log N)$ repetitions suffice to determine r but there is still a significant gap in our argument: the Fourier transform \mathcal{F} that we used is a large non-trivial unitary operation, of size N by N, and we cannot ab initio just assume that it can be implemented using only poly $(\log N)$ basic computational operations. Indeed it may be shown that any d by d unitary operation may be implemented on a quantum computer (equipped with any universal set of operations) in $O(d^2)$ steps [8]. This is also the number of steps needed for the classical computation of multiplying a d by d matrix into a d dimensional column vector. For our use of \mathcal{F} this bound of $O(N^2)$ does not suffice. Fortunately the Fourier transform (FT) has extra special properties which enable it to be implemented in $O((\log N)^2)$ steps. These properties stem from the classical theory of the fast Fourier transform (FFT) [12] which shows how to reduce the $O(N^2)$ steps of classical matrix multiplication to $O(N \log N)$ steps. If the same ideas are implemented in a quantum setting then it may be seen [8, 11] that the number of steps is reduced to $O((\log N)^2)$ giving our desired implementation. Note also that according to eq. (4) we have

$$\mathcal{F}\left|0\right\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \left|x\right\rangle$$

so that once we have an efficient implementation of \mathcal{F} we will be able to efficiently produce the uniform large superposition in the input register, necessary to get $|f\rangle$ in eq. (2).

The technical details of the efficient implementation of FT are given in §3 of [11] but the essential idea is the following. We will be able to efficiently implement FT in dimensions which are powers of 2 rather than arbitrary N. Thus we use the smallest power of 2 that is larger than N. (In later applications this slight mismatch of dimensions can be shown to not cause problems, although the rigorous demonstration of this [7] can become technically complicated). Let n denote the least integer greater than $\log_2 N$. Then the required Fourier transform FT is a unitary operation on n qubits. The FFT formalism gives an explicit way of decomposing FT on n qubits into a sequence of gates where each gate acts on at most two qubits and the length of the sequence is polynomial in n (actually $O(n^2)$). FT is a very special operation in this regard – a general unitary operation would require a sequence of exponential length! Consider now the action of a 2-qubit gate U on a state $|\alpha\rangle$ of n qubits. Suppose that U acts on the first two qubits and that U has matrix elements $U_{j_1j_2}^{i_1i_2}$ in a standard product basis of the n qubit state space. Suppose that $|\alpha\rangle$ has components $a_{i_1...i_n}$ (where all indices range over the values 0 and 1). The components of the updated state are given by matrix multiplication:

$$a_{i_1...i_n}^{new} = \sum_{j_1,j_2} U_{i_1i_2}^{j_1j_2} a_{j_1j_2i_3...i_n}.$$
 (10)

This update counts as one step of quantum computation (or more precisely a constant number, independent of n to implement U) and the FFT decomposition amounts to an implementation of FT in $O(n^2)$ steps on a quantum computer. In contrast if eq. (10) is viewed as a classical computation, we must perform a 4×4 matrix multiplication 2^{n-2} times (for all values of the

string $i_3 ... i_n$). This ultimately gives an implementation of FT with $O(n2^n)$ classical steps, which is the standard fast Fourier transform algorithm.

In summary, the quantum algorithm for determining the periodicity of a given function f, with N inputs, begins with the computation of all values of f in superposition using one application of FT and one evaluation of f. FT is then applied to pick out the periodic structure of the resulting state. The quantum implementation of the FFT algorithm guarantees that FT may be implemented in $\operatorname{poly}(\log N)$ steps. An analogous classical computation would require O(N) invocations of f to compute a column vector of all the function values and then $O(N \log N)$ steps to perform the FFT. Thus the quantum algorithm represents an exponential speedup.

3 Quantum factoring

The problem of integer factorisation is the following: given a number N, of $n = \log_2 N$ digits, we wish to determine a number k (not equal to 1 or N) which divides N exactly. We now outline how this problem may be reduced to a problem of periodicity determination for a suitable periodic function f. Then the quantum algorithm described in the preceding section will achieve the factorisation of N in poly(n) time i.e. polynomial in the number of digits of N.

We note first that there is no known classical algorithm which will factorise any given N in a time polynomial in the number of digits of N. For example the most naive factoring algorithm involves test-dividing N by each number from 1 to \sqrt{N} (as any composite N must have a factor in this range). This requires at least \sqrt{N} steps (at least one step for each trial factor) and $\sqrt{N} = 2^{\frac{1}{2}n}$ is exponential in n. In fact using all the ingenuity of modern mathematics, the fastest known classical factoring algorithm runs in a time of order $\exp(n^{\frac{1}{3}}(\log n)^{\frac{2}{3}})$.

To reduce the problem to a problem of periodicity we will need to use some basic results from number theory. These are further described in the appendix of [8] and complete expositions may be found in most standard texts on number theory such as [1, 2]. We begin by selecting a number a < N at random. Using Euclid's algorithm, we compute in poly(log N) time, the highest common factor of a and N. If this is larger than 1, we will have found a factor of N and we are finished! However it is overwhelmingly likely that a randomly chosen a will be coprime to N (e.g. if N is the product of two large primes). If a is coprime to N, then Euler's theorem of number theory guarantees that there is a power of a which has remainder 1 when divided by N. Let r be the smallest such power:

$$a^r \equiv 1 \mod N$$
 and r is the least such power (11)

(If a is not coprime to N then no power of a has remainder 1). r is called the *order* of a modulo N. Next we show that the information of r can provide a factor of N.

Suppose that we have a method for determining r (c.f. later) and suppose further that r comes out to be an *even* number. Then we can rewrite eq. (11) as $a^r - 1 \equiv 0 \mod N$ and factorise as a difference of squares:

$$(a^{r/2} - 1)(a^{r/2} + 1) \equiv 0 \mod N$$
(12)

Let $\alpha = a^{r/2} - 1$ and $\beta = a^{r/2} + 1$. Then N exactly divides the product $\alpha\beta$. If neither α nor β is a multiple of N then N must divide partly into α and partly into β . Thus computing

the highest common factor of N with α and β (again using Euclid's algorithm) will generate a non-trivial factor of N.

As an example take N=15 and choose the coprime number a=7. By computing the powers of 7 modulo 15 we find that $7^4 \equiv 1 \mod 15$ i.e. the order of 7 modulo 15 is 4. Thus 15 must exactly divide the product $(7^{4/2}-1)(7^{4/2}+1)=(48)(50)$. Computing the highest common factor of 15 with 50 and 48 gives 5 and 3 respectively, which are indeed nontrivial factors of 15.

Our method will give a factor of N provided that r comes out to be even and that neither of $(a^{r/2} \pm 1)$ are exact multiples of N. To guarantee that these conditions occur often enough (for randomly chosen a's) we have

Theorem: Let N be odd and suppose that a < N coprime to N is chosen at random. Let r be the order of a modulo N. Then the probability that r is even and $a^{r/2} \pm 1$ are not exact multiples of N is always $\geq \frac{1}{2}$.

The (somewhat lengthy) proof of this theorem may be found in appendix B of [8], to which we refer the reader for details.

Overall, our method will produce a factor of N with probability at least half in every case. This success probability may be amplified as close as desired to 1, since K repetitions of the procedure (with K constant independent of N) will succeed in factorising N with probability exceeding $1 - \frac{1}{2K}$.

All steps in the procedure, such as applying Euclid's algorithm and the arithmetic manipulation of numbers, can be done in poly(n) time. The only remaining outstanding ingredient is a method for determining r in $poly(\log N)$ time. Consider the exponential function:

$$f(x) = a^x \mod N \tag{13}$$

Now eq. (11) says precisely that f is periodic with period r i.e. that f(x+r) = f(x). Thus we use the quantum algorithm for periodicity determination, described in the previous section, to find r. To apply the algorithm as stated, we need to restrict the scope of x values in eq. (13) to a finite range $0 \le x \le q$ for some q. If q is not an exact multiple of (the unknown) r i.e. q = Ar + t for some 0 < t < r, then the resulting function will not be exactly periodic—the single final period over the last t values will be incomplete. However if q is chosen large enough, giving sufficiently many intact periods of f, then the single corrupted period will have negligible effect on the use of the q by q Fourier transform to determine r, as we might intuitively expect. In fact it may be shown that if q is chosen to have size $O(N^2)$ then we get a reliable efficient determination of r. For the technical analysis of this imperfect periodicity (involving the theory of continued fractions) we refer the reader to [7, 8]. q is also generally chosen to be a power of 2 to allow an efficient implementation of FT via the FFT formalism.

4 Evaluation of discrete logarithms

In the previous section we showed how the problem of factoring may be reduced to a question of periodicity of a function on \mathcal{Z}_N , the additive group of integers modulo N. We now introduce the problem of discrete logarithms and show how it may also be reduced to a slightly more general kind of periodicity – on the additive group of pairs of integers modulo N. These important special cases provide the basis for the generalization in the next section to an elegant and natural group theoretic setting.

Let p be a prime number and let \mathcal{Z}_p^* denote the group of integers $\{1, 2, \dots, p-1\}$ under

multiplication modulo p. Note that for general values of m the set $\mathcal{Z}_m^* = \{1, 2, \dots, m-1\}$ is not a group under multiplication modulo m as we do not generally have multiplicative inverses (e.g. in \mathcal{Z}_6 there is no number x satisfying $3x \equiv 1 \mod 6$ i.e. 3 has no inverse) but if p is prime then \mathcal{Z}_p^* is always a group.

A number g in \mathbb{Z}_p^* is called a generator (or primitive root mod p) if the powers of g generate all of \mathbb{Z}_p^* i.e. $\mathbb{Z}_p^* = \{g^0 = 1, g^1, g^2, \dots, g^{p-2}\}$. (For example in \mathbb{Z}_5^* 2 and 3 are generators but 1 and 4 are not). Thus every element x of \mathbb{Z}_p^* may be written uniquely as $x = g^y$ for some y in \mathbb{Z}_{p-1} . y is called the discrete logarithm of x (with respect to g) and we write $y = \log_g x$. Note that multiplication of x's mod p corresponds to addition of y's mod (p-1) so a generator provides a way of identifying \mathbb{Z}_p^* as \mathbb{Z}_{p-1} .

The problem of discrete logarithms is the following: we have p and a generator g of \mathbb{Z}_p^* . For any $x \in \mathbb{Z}_p^*$ we want to compute its discrete logarithm $y = \log_g x$. Let n be the number of digits of p. The fastest known classical algorithm runs in time of order $\exp(n^{\frac{1}{3}}(\log n)^{\frac{2}{3}})$ whereas our quantum algorithm will run in time less than $O(n^3)$.

We begin by noting that multiplicative inverses in \mathbb{Z}_p^* may be computed efficiently using Euclid's algorithm. Indeed for any x we have the highest common factor of x and p being 1 so Euclid's algorithm provides integers a and b such that ax + bp = 1 so $ax \equiv 1 \mod p$ and a is the desired inverse.

Consider $G = \mathcal{Z}_{p-1} \times \mathcal{Z}_{p-1}$, the additive group of pairs of integers and for given x, g, p, the function $f : \mathcal{Z}_{p-1} \times \mathcal{Z}_{p-1} \to \mathcal{Z}_p^*$ given by

$$f(a,b) = g^a x^{-b} \bmod p$$

which is computable in time poly(n). In terms of the discrete logarithm $y = \log_q x$ we have

$$f(a,b) = g^{a-yb} \bmod p$$

so

$$f(a_1, b_1) = f(a_2, b_2)$$
 if and only if $(a_2, b_2) = (a_1, b_1) + \lambda(y, 1)$ for $\lambda \in \mathcal{Z}_{n-1}$.

Thus the pair (y, 1) is the period of f on its product domain. To determine y our quantum algorithm will follow the standard period–finding procedure of section 2, slightly generalized to deal with the fact that the domain consists of pairs rather than just single numbers.

We consider a Hilbert space with an orthonormal basis $\{|a\rangle|b\rangle: a,b\in\mathcal{Z}_{p-1}\}$ labeled by the elements of G and begin by computing an equal superposition of all values of f:

$$|f\rangle = \frac{1}{p-1} \sum_{a,b} |a\rangle |b\rangle |f(a,b)\rangle.$$

If we measure the last register and see a value $k_0 = f(a_0, b_0)$ we obtain the periodic state

$$|\psi\rangle = \frac{1}{\sqrt{p-1}} \sum_{k=0}^{p-2} |a_0 + ky\rangle |b_0 + k\rangle.$$

To eliminate the dependence of the labels on the randomly chosen (a_0, b_0) we apply \mathcal{F} , the Fourier transform modulo (p-1) to each of the two registers. The calculations are very similar to those for factoring (c.f. eq. (8)). Let us introduce the functions

$$\chi_{l_1,l_2}(a,b) = \exp 2\pi i \left(\frac{al_1 + bl_2}{p-1}\right).$$

Then (similar to eq. (7)) $\mathcal{F} \otimes \mathcal{F} |\psi\rangle$ will yield an equally weighted superposition of those labels (l_1, l_2) such that $\chi_{l_1, l_2}(y, 1) = 1$ i.e. $yl_1 + l_2 \equiv 0 \mod p - 1$ so $l_2 = -yl_1 \mod p - 1$ and $l_1 = 0, 1, \ldots, p-2$. Explicitly we have

$$\mathcal{F} \otimes \mathcal{F} |\psi\rangle = \frac{1}{\sqrt{p-1}} \sum_{l_1=0}^{p-2} \exp 2\pi i \left(\frac{a_0 l_1 - b_0 y l_1}{p-1}\right) |l_1\rangle |-y l_1\rangle.$$

Then a measurement of the labels will provide a pair $(l_1, l_2) = (l_1, -yl_1 \mod p - 1)$ where $l_1 \in \mathcal{Z}_{p-1}$ is chosen uniformly at random. If l_1 happens to be coprime to p-1 we can use Euclid's algorithm to find l_1^{-1} , the multiplicative inverse modulo p-1, and compute y as $-l_1^{-1}l_2$. If l_1 is not coprime to p-1 then we cannot uniquely determine y from (l_1, l_2) . What is the probability that a uniformly chosen l_1 is coprime to p-1? In section 2 we saw that this probability will be of order $1/\log\log(p-1)$ and so to determine y with high probability we will need to repeat our algorithm a very modest $O(\log\log p)$ times (which is even exponentially smaller than our goal of poly $(\log p)$ times).

As in the case of factoring there is the residual issue of efficiently implementing the Fourier transform that is used. To take advantage of the FFT formalism we would want to use FT for integers modulo a power of 2 (instead of modulo p-1). Let 2^t be the smallest power of 2 greater than p-1, so t is the smallest integer greater than $\log_2(p-1)$. Then FT modulo 2^t may be implemented in $O(t^2) = O((\log p)^2)$ steps. If we use FT modulo 2^t in place of FT modulo p-1 in the above algorithm then we will obtain a larger set of possible output pairs (l_1, l_2) with varying probabilities. However as in the case of factoring, these pairs will lie with high probability sufficiently near to the "good" pairs $(l_1, -yl_1)$ where l_1 is coprime to p-1, so that p may still be determined. The details of dealing with the nearby pairs and assessing their probabilities, are quite involved and given in [7].

5 The abelian hidden subgroup problem

Given the above developments it is exciting to observe that the concept of periodicity and the construction of the Fourier transform may be generalized to apply to any finite group G. Our discussion so far pertains simply to the special cases of the additive group of integers modulo N (for factoring) and the product group $\mathcal{Z}_{p-1} \times \mathcal{Z}_{p-1}$ (for evaluating discrete logarithms). The generalized viewpoint will also provide considerable insight into the workings of the Fourier transform. We will now outline the essential ideas involved restricting attention in this section to the case of finite abelian groups.

Let G be any finite abelian group. Let $f: G \to X$ be a function on the group (taking values in some set X) and consider

$$K = \{k \in G : f(k+q) = f(q) \text{ for all } q \in G\}$$
 (14)

(Note that we write the group operation in additive notation). K is necessarily a subgroup of G called the stabilizer or symmetry group of f. It characterizes the periodicity of f with respect to the group operation of G. For factoring where G was \mathcal{Z}_N , K was the cyclic subgroup of all multiples of r.

The condition (14) is equivalent to saying that f is constant on the cosets of K in G. (Recall that the cosets are subsets of G of the form $g + K = \{g + k : k \in K\}$ and they partition all of G into disjoint parts of equal size |K|).

Given a device that computes f, our aim is to suitably determine the "hidden subgroup" K e.g. we may ask for a set of generators for K or for an algorithm that outputs a randomly chosen element of K. More precisely we wish to obtain this information in time $O(\text{poly}(\log |G|))$ where |G| is the size of the group and the evaluation of f on an input counts as one computational step. (Note that we may easily determine K in time O(poly(|G|)) by simply evaluating and examining all the values of f). We begin as in our examples by constructing the state

$$|f\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |f(g)\rangle$$

and read the second register. Assuming that f is suitably non-degenerate – in the sense that $f(g_1) = f(g_2)$ iff $g_1 - g_2 \in K$ i.e. that f is one-to-one within each period – we will obtain in the first register

$$|\psi(g_0)\rangle = \frac{1}{\sqrt{|K|}} \sum_{k \in K} |g_0 + k\rangle \tag{15}$$

corresponding to seeing $f(g_0)$ in the second register and g_0 has been chosen at random. In eq. (15) we have an equal superposition of labels corresponding to a randomly chosen coset of K in G. Now G is the disjoint union of all the cosets so that if we read the label in eq. (15) we will see a random element of a random coset, i.e. a label chosen equiprobably from all of G, yielding no information at all about K.

The general construction of a "Fourier transform on G" will provide a way of eliminating g_0 from the labels (just as in the case of \mathcal{Z}_N) and the resulting state will then provide direct information about K. Let \mathcal{H} be a Hilbert space with a basis $\{|g\rangle:g\in G\}$ labeled by the elements of G. Each group element $g_1\in G$ gives rise to a unitary "shifting" operator $U(g_1)$ on \mathcal{H} defined by

$$U(g_1)|g\rangle = |g + g_1\rangle$$
 for all g

For any coset $g_0 + K$ let us write $|g_0 + K\rangle$ for the uniform superposition $\frac{1}{\sqrt{|K|}} \sum_{k \in K} |g_0 + k\rangle$. Note that the state in eq. (15) may be written as a g_0 -shifted state:

$$|g_0 + K\rangle = U(g_0)|K\rangle \tag{16}$$

Our basic idea now is to introduce into \mathcal{H} a new basis $\{|\chi_g\rangle:g\in G\}$ of special states which are *shift-invariant* in the sense that

$$U(g_1) |\chi_{g_2}\rangle = e^{i\phi(g_1, g_2)} |\chi_{g_2}\rangle$$
 for all g_1, g_2

i.e. the $|\chi_g\rangle$'s are the common eigenstates of all the shifting operations U(g). Note that the U(g)'s all commute (since the group is abelian) so such a basis of common eigenstates is guaranteed to exist. Then according to eq. (16) if we view $|K\rangle$ and $|g_0 + K\rangle$ in the new basis, they will contain the same pattern of labels determined by the subgroup K only, and corresponding amplitudes will differ only by phase factors. Thus the probability distribution of the outcomes of a measurement in the new basis will directly provide information about the subgroup K. More precisely it may be shown [11] (and cf below) that this measurement provides a uniform random sample from the so-called dual group of K in G.

The Fourier transform \mathcal{F} on G is defined to simply be the unitary transformation which rotates the shift invariant basis back to the standard basis:

$$\mathcal{F}|\chi_g\rangle = |g\rangle$$
 for all g

Hence to read $|\psi(g_0)\rangle$ in the new basis we just apply \mathcal{F} and read in the standard basis.

To give an explicit construction of \mathcal{F} it suffices to give the states $|\chi_g\rangle$ written as components in the standard basis. There is a standard way of calculating these components based on constructions from group representation theory. An introduction with further references is given in [10, 11] and here we will summarize the main points. If we write

$$|\chi_l\rangle = \frac{1}{\sqrt{|G|}} \sum_{q} \chi_l(g) |g\rangle \quad \text{for each } l \in G$$
 (17)

then we can take the functions $\chi_l: G \to \mathcal{C}$ to be the |G| irreducible representations of the group G. Then the basic theorems of group representation theory (cf for example [10]) guarantee that the states $|\chi_l\rangle$ are orthonormal and have the required shift invariant property. Indeed shift invariance is a direct consequence of the basic defining property of a representation: $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$. For the group \mathcal{Z}_N the irreducible representations are given by $\chi_k(j) = \exp 2\pi i \, jk/N$ for $j, k \in \mathcal{Z}_N$ and

$$|\chi_k\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i \frac{jk}{N}} |j\rangle$$

leading to the Fourier transform formula given in eq. (4).

Which labels l appear in $\mathcal{F}|g_0 + K\rangle$? It suffices to consider $\mathcal{F}|K\rangle$ and from eq. (17) we get directly

$$\mathcal{F}|K\rangle = \frac{1}{\sqrt{|G|}\sqrt{|K|}} \sum_{l \in G} \left(\sum_{k \in K} \chi_l(k) \right) |l\rangle$$

Now, for Abelian groups, the restriction of χ_l from G to K is an irreducible representation of K and the orthogonality relations for irreducible representations give that $\sum_{k \in K} \chi_l(k) = 0$ for all χ_l 's except the trivial representation defined by $\chi_l(k) = 1$ for all $k \in K$. In the latter case we have $\sum_{k \in K} \chi_l(k) = |K|$. Hence $\mathcal{F}|K\rangle$ is a uniform superposition of the |G|/|K| labels l such that χ_l restricts to the trivial representation on K. If K has a generator r then the latter condition is equivalent to $\chi_l(r) = 1$ as we saw in the example of factoring and discrete logarithms (where r = (y, 1)). Thus we are able to uniformly sample from this set of labels, which distinguishes the possible K's. This completes the quantum part of the algorithm but to convert this into an explicit description of K (say an actual set of generators) we need to use further mathematical properties of G e.g. properties of co-primality as illustrated in our examples.

The above group-theoretic framework serves to generalize and extend the applicability of the quantum algorithm for periodicity determination. For example Simon considered the following problem: suppose that we have a black box which computes a function f from n-bit strings to n-bit strings. It is also promised that the function is "two-to-one" in the sense that there is a fixed n-bit string ξ such that

$$f(x+\xi) = f(x)$$
 for all *n*-bit strings *x*. (18)

(Here + denotes binary bitwise addition of n bit strings.) Our problem is to determine ξ .

To see that this is just a generalized periodicity determination note that in the group $(\mathcal{Z}_2)^n$ of *n*-bit strings, every element satisfies x+x=0. Hence eq. (18) states just that f is periodic on the group with periodicity subgroup $K=\{0,\xi\}$. Thus to determine ξ we construct the

Fourier transform on the group of *n*-bit strings and apply the standard algorithm above. The relevant Hilbert space \mathcal{H} with a basis labeled by *n*-bit strings is just a row of *n* qubits. The irreducible representations of the group \mathcal{Z}_2^N are the functions $f_x(y) = (-1)^{x_1y_1} \dots (-1)^{x_ny_n}$ where $x = x_1 \dots x_n$ and $y = y_1 \dots y_n$ are n bit strings. Thus the Fourier transform may be easily seen [11] to be just the application of the 1-qubit Hadamard transform:

$$H = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)$$

to each of the n qubits. The resulting quantum algorithm for determining the hidden subgroup then reproduces Simon's original algorithm [6]. It determines ξ in $O(n^2)$ steps whereas it may be argued [6] that any classical algorithm must evaluate f at least $O(2^n)$ times.

6 Non-abelian groups

We will now consider the hidden subgroup problem in the situation where G and the subgroup K may be non-abelian i.e. we have $f: G \to X$ which is constant on the (left) cosets of K in G. We now also write the group operation multiplicatively. As before our algorithm begins in the same way by producing the state $|g_0K\rangle$ where g_0 has been chosen at random. The passage from abelian to non-abelian groups is accompanied by various potential conceptual problems:

(a) (Construction of non-abelian Fourier transform). For abelian groups the irreducible representations are always one dimensional (i.e. the functions χ_l in eq. (17)) whereas for non-abelian groups they are functions $\chi: G \to U(d)$ taking values in the set U(d) of all $d \times d$ unitary matrices for suitable values of d. According to a basic theorem of group representation theory [3], if $d_1 \dots, d_m$ are the dimensions of a complete set of irreducible unitary representations χ_1, \dots, χ_m then $d_1^2 + \dots + d_m^2 = |G|$. Let us write $\chi_{i,jk}(g)$ for the (j,k)th component of the unitary matrix $\chi_i(g)$. Thus as i,j,k vary we get |G| complex valued functions and as in eq. (17) we may define the |G| states:

$$|\chi_{i,jk}\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \chi_{i,jk}(g) |g\rangle.$$

The orthogonality relations of irreducible representations [3] guarantee that these are again orthonormal states, called the Fourier basis, and the non-abelian Fourier transform is defined as the unitary operation that rotates this basis into standard position. In the abelian case, j and k take only the value 1 and may be omitted. The Fourier basis may be grouped into m subsets of sizes d_1^2, \ldots, d_m^2 according to the value of i and we may consider the associated incomplete von Neumann measurement which distinguishes only the various representations. We will denote this incomplete measurement by \mathcal{M}_{rep} and it will be important later (cf (d) below).

(b) (Efficient implementation of non-abelian FT). For the efficiency of our quantum algorithms it is important that FT be implementable in poly(log |G|) computational steps. In the abelian case this was a consequence of the FFT formalism. Fortunately this formalism extends to the non-abelian case too [12] requiring only that the group contains a suitable tower of subgroups. For the standard FFT on \mathbb{Z}_{2^n} this tower is $H_0 \subset H_1 \subset \ldots \subset \mathbb{Z}_{2^n}$ where H_k is the subgroup of multiples of 2^{n-k} in \mathbb{Z}_{2^n} . A fundamental non-abelian group is the

permutation group $G = \mathcal{P}_n$ on n symbols. \mathcal{P}_n contains the tower $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_n$ and its FT has been shown to be efficiently implementable [13].

- (c) (Description of subgroups). Our quantum algorithm should provide distinguishable outputs for different possible subgroups K. In that case we say that the subgroup has been information-theoretically determined. However in general it may still be a difficult computational task to identify the actual subgroup from the output result. For finite abelian groups a fundamental structure theorem [4] asserts that any such group is isomorphic to a direct product of groups of the form \mathcal{Z}_n . In this case any subgroup K will have a simple poly($\log |G|$) sized description given by a list of generators, which we can require as the output of the algorithm. For non-abelian groups the classification of possibilities is not so simple. For example even the problem of deciding whether or not two sets of generators and relations give isomorphic groups, is known to be uncomputable! [4]. Furthermore it is not appropriate to ask for a list of all elements of K as this may be of size O(|G|) i.e. exponentially large in $\log |G|$. We may circumvent these difficulties of description by asking for less instead of characterising K per se, we may for example ask that the algorithm outputs a randomly chosen element of K or determines whether or not some chosen property of a subgroup holds for the hidden subgroup.
- (d) (Shift invariance). In the preceding section we used the existence of the shift invariant basis $|\chi_l\rangle$ to give some intuitive insight into why FT is useful for abelian hidden subgroups. It provided a means of eliminating the effects of a randomly chosen g_0 in the state $|g_0 + K\rangle$. The existence of a shift invariant basis relies on the commuting of the shift operators U(q) and this is a consequence of the abelian-ness of G. In the non-abelian case such a basis will not exist. However a restricted form of shift invariance still survives because of the multiplicative property of representations: $\chi_i(g_1g_2) = \chi_i(g_1)\chi_i(g_2)$ (where the RHS is multiplication of $d_i \times d_i$ unitary matrices). If we perform a complete measurement for the labels i, j, k (as in (a) above) on the state $|gK\rangle$ then the resulting probability distribution will not be independent of g. However if we perform the incomplete measurement \mathcal{M}_{rep} then it is a simple consequence [5] of the above multiplicative property that the outcome distribution is independent of q, providing direct (but generally incomplete) information about K itself. (In the abelian case this distribution is the uniform distribution over the dual group of K in G). In a similar way if K and L are conjugate subgroups (i.e. $L = g_0 K g_0^{-1}$ for some g_0) then any coset states $|g_1K\rangle$ and $|g_2L\rangle$ will also give identical output distributions and hence the measurement \mathcal{M}_{rep} cannot distinguish conjugate subgroups. (In the abelian case this is not a problem since subgroups are conjugate if and only if they are equal).

There is no known efficient quantum algorithm that will solve the hidden subgroup problem in general but we have various significant partial results.

Let G be any finite group and assume that the FT on G can be efficiently computed. Under this assumption, Hallgren, Russell and Ta-Shma [5] have shown that the hidden subgroup problem may be efficiently solved for any normal subgroup K of G. We proceed as usual by first constructing a randomly chosen coset state $|g_0K\rangle$ (as in section 4) and then performing the measurement \mathcal{M}_{rep} in (a) (by performing FT and reading the representation labels i only). It is shown in [5] that K may be reconstructed with high probability from $O(\log |G|)$ repetitions of this procedure i.e. the $O(\log |G|)$ measurement outcomes determine K information theoretically.

For abelian groups G (where all subgroups are normal) this would solve the general abelian hidden subgroup problem, except that FT cannot be exactly implemented efficiently for a general abelian G. Recall that in the examples of factoring and discrete logarithms

we needed to replace the Fourier transform by a slightly larger one – in a dimension that was a power of 2 – to take advantage of the FFT formalism. This approximation to the true FT on G was sufficiently close to still allow the determination of the abelian hidden subgroup. Kitaev [9] has described similar efficient approximations to the FT on any abelian group which should suffice for our purposes. Also, in view of (c) above, we could ask that the algorithm in the abelian case determines K more explicitly – by outputting an actual set of generators as in the examples of factoring and discrete logarithms. Again this should be possible but the detailed description of an efficient quantum algorithm for the general abelian hidden subgroup problem seems not to have been described in the literature (although the essential ingredients appear to be implicit in the work of Kitaev [9] and Shor's treatment [7] of factoring and discrete logarithms).

Returning to the most general hidden subgroup problem, Ettinger, Hoyer and Knill [14] have shown that $N = O(\log |G|)$ preparations of random coset states $|g_1K\rangle, \ldots, |g_NK\rangle$ always suffice to determine K information theoretically i.e. there exists a quantum observable on the state $|g_1K\rangle \otimes |g_2K\rangle \otimes \ldots \otimes |g_NK\rangle$ which will distinguish all possible K's with high probability (for any random choices of g_1, \ldots, g_N). However it is not known how to efficiently implement such an observable in general. For the special case of normal K's the result of Hallgren, Russell and Ta-Shma gives precisely such an efficiently implementable observable.

To conclude we will describe an important open question which can be formulated as a non-abelian hidden subgroup problem. This is the so-called graph isomorphism problem.

An (undirected) graph A with n vertices labeled 1, 2, ..., n may be described by an n by n matrix M_A with entries that are either 0 or 1. The ij^{th} entry is 1 if and only if the graph has an edge joining vertices i and j (and we assume that A always has at most one edge joining two vertices). Let \mathcal{P}_n denote the group of all permutations of n symbols 1, 2, ..., n. Two graphs A and B are said to be isomorphic if B can be made identical to A by a relabeling of its vertices i.e. if there exists a permutation $\Pi \in \mathcal{P}_n$ such that M_A is obtained by simultaneously permuting the rows and columns of M_B by Π . The symmetry group of any graph A on n vertices is the subgroup of all permutations Π which leave M_A unchanged when Π is applied to the rows and columns simultaneously. The graph isomorphism problem is the following: given two connected graphs A and B, each on n vertices, determine whether they are isomorphic or not. We wish to perform this efficiently i.e. in poly(n) steps. There is no known efficient classical solution.

To re-formulate this problem as a hidden subgroup problem, let C be the graph which is the disjoint union of A and B, having 2n vertices labeled $1, 2, \ldots, n, n+1, \ldots, 2n$ where $1, 2, \ldots, n$ label A and $n+1, \ldots, 2n$ label B. The symmetry group K of C is evidently a subgroup of \mathcal{P}_{2n} but we can say more: since A and B are connected and C is the disjoint union, any symmetry of C must either separately permute the sets of labels $L_A = \{1, 2, \ldots, n\}$ and $L_B = \{n+1, \ldots, 2n\}$ or else swap the two sets entirely. Thus if H denotes the group $\mathcal{P}_n \times \mathcal{P}_n$ and σ is the permutation of $1, 2, \ldots, 2n$ that swaps the two sets S_A and S_B in their listed order, then K will always be a subset of the group $G = H \cup \sigma H$. H is the subgroup of G containing all permutations that map S_A and S_B into themselves whereas σH is its one other coset, of all permutations that swap the elements of S_A and S_B (in some arbitrary order). Now we may easily verify the following facts:

- (i) if A and B are not isomorphic then K lies entirely in H,
- (ii) if A and B are isomorphic then exactly half of the members of K are in H and half are in σH .

Given any element $\Pi \in G$ it is easy to check whether it lies in H or σH (e.g. we just

compute $\Pi(1)$ and check whether it is $\leq n$ or $\geq n+1$). Hence we will have efficiently solved the graph isomorphism problem if we are able to randomly sample from the elements of K. This is a weak form of the hidden subgroup problem in which we are not asking for the full information of K but merely whether it overlaps σH by half of its elements or is disjoint from σH , knowing that one of these two must always holds. In our standard algorithm the function f used to generate the random coset state $|g_0K\rangle$ is the efficiently computable $f: G \to X$ where X is the set of all matrices of size $2n \times 2n$ with 0,1 entries and $f(\Pi)$ is the matrix obtained by permuting the rows and columns of M_C by Π .

Unfortunately none of the known partial results about efficient quantum algorithms for determining hidden subgroups seem to apply to this formulation of the graph isomorphism problem and the possibility of an efficient solution remains an open challenge. However given the already demonstrated success and mathematical elegance of the Fourier transform formalism we can be optimistic that an efficient algorithm might be derived along these lines.

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